# ARTICLE XI.

*On a new Principle in regard to the Power of Fluids in Motion to produce Rupture of the Vessels which contain them; and on the Distinction between accumulative and instantaneous Pressure. By Charles Bonnycastle, Professor of Mathematics in the University of Virginia. Read November* 15, 1840.

In a paper published by Dr. Hare in the Transactions of the American Philosophical Society, of which paper he was polite enough to send me a copy, is a description of a singular phenomenon that was observed in the destruction of an air-vessel, under circumstances which at first appear to contradict the known laws that regulate the motion and pressure of fluids.

The subject of the paper is "The collapse of a reservoir whilst apparently subject within to great pressure from a head of water;" and in treating it, Dr. Hare points out the attendant circumstances, and ingeniously shows how the vessel may have been momentarily relieved from the pressure of the water within, so as to make the pressure of the surrounding air efficient in producing the collapse. He does not, however, push his inquiries to a point essential to the full explanation. An investigation into the nature and degree of the forces brought into action leads to results, according, indeed, with the theory of resilience, but presenting, in the laws of instantaneous pressure, and in the effects of that pressure when produced by an abrupt check given to a fluid in motion, a branch of the subject in some degree intermediate between statics and dynamics, and which has not yet been fully developed.

At the period when I first deduced the results contained in this paper, very little attention had been paid to any of the class of phenomena to which they belong; something further has since been done in England, but I have not seen

any investigations that bear directly on the subject in question. My own remarks were drawn up immediately after the publication of the memoir above noticed, but were laid by and neglected until my attention was again drawn towards them by an instance of the great effect of such pressures that fell under my notice a few months since. I had occasion to trace the laws of sound propagated through water, and with the view of readily discharging a pistol at various depths below the surface, contrived the apparatus in Fig. 1. I had directed the box to be formed of cast iron, but the maker, providing a strong support for the breech of the pistol, and meeting with delay in the casting, persuaded himself that thick tin plate, strengthened with ribs, would give abundant strength to the remaining sides of the box. A covering of sheet brass, an eighth of an inch thick, was subsequently added, and yet such is the nature of the action in question, that farther security was necessary before this little case, which did not exceed eight inches by five, could resist the powerful strains that tended to burst it inward when exposed to the action of an ordinary pistol, exploded at a depth of five or six feet below water.

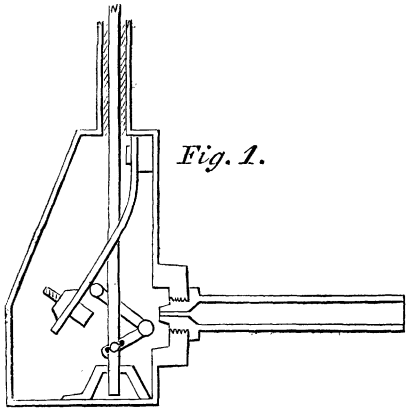


*Fig. 1.*

This additional instance of the occurrence of those destructive and apparently paradoxical strains that had attracted the attention of the eminent chemist above mentioned, induced me to re-examine the paper I formerly showed to you,* and to make public, with some slight addition, the results it contained.

I had shown, in a former paper, that many problems which appear purely questions of statics, cannot be resolved without employing the elementary motions which connect such problems with those of dynamics.† The problem before us is one of this class, and, when treated practically, is most conveniently arranged as a question of statical pressures.

Practical mechanics and engineers are accustomed to view the forces with which they have to deal, under the twofold division of the *loads* which their engines or structures bear, and the *strains* such loads occasion.

* This paper was addressed to Dr. Patterson.

† On the pressures produced by a heavy body when sustained by more than three supports. Read before the R. Society of London, 1819. An. Philos., April, 1819.

The pressures that maintain the equilibrium between the load and strain may vary incessantly from the operation of those minute vibrations that every where render absolute rest impossible; nearly all the cases that are practically regarded as states of rest, or uniform motion, are really examples of periodic vibrations which escape the senses. Motions so small are often sufficient to make the pressures pass through all degrees of intensity, and even to become negative; but the practical mathematician, who only looks to ultimate results, cannot take notice of this delicate and hidden machinery, and finds it convenient to regard the load as invariable, and to estimate the strain, not by its mean, but its greatest amount. These remarks will sufficiently explain the necessity of the distinction which I have made between the pressure and the mean pressure, or load: and I have only to add one or two farther observations, in order to render the terms I have employed sufficiently clear and precise. The first of these regards a species of action that occurs frequently in the strains we are discussing, or rather, which forms a convenient limit to the mixed strains that are met with in nature. To understand this action, let us suppose a fluid of finite mass, as water or mercury, to be contained in a tube of some very flexible and imponderable material; a sudden tremor propagated in the fluid would then cause, at the point where the wave passed, a pulsation of very limited extent, compared with the expansive power of the tube; and which would have its single advance and single regression regulated by laws proper to the fluid, and little involving the inconsiderable resistance of the conduit. Such an action we shall describe as "an exterior, or interior, vibration of immense momentum and infinitely little extent, compared with the motions and mass of the support."

A clear idea of this species of force will be necessary to what follows; and the only further observation for which we shall have occasion has reference, not to the forces exerted, but to the nature of the structure which sustains them; and chiefly to the distinction that must be made between elastic supports and those which unite with this quality a large degree of flexibility. Every material, it is well known, when wrought into rods or plates sufficiently thin, loses its tendency to return to a primary, or unloaded position, and must be regarded as flexible. Flexibility itself is merely an elasticity, more or less perfect, acting according to a single axe; and flexible bodies may be allied to the

elastic by regarding them as secured at each end and drawn into an initial position by the action of an evanescent stretching force.

From which observations it is evident that if we consider the several cases of perfect and imperfect elasticity in the directions of all the axes, and of perfectly flexible bodies, drawn into an initial state by the action of an initial and evanescent force, we shall obtain the limits of the results that can practically happen.

The several conclusions to which I have arrived for these cases may be stated as follows:—

1. It is convenient to distinguish between accumulative and instantaneous loads, or between those which are gradually increased until the deflection due to the ultimate load is obtained, and those which commence in their full efficacy from the initial position of the support.

2. Within the limits of perfect elasticity, instantaneous pressure produces twice the effect of that which is accumulative, whether the result be to produce deflection or fracture.

3. In regard to supports perfectly elastic in one direction and perfectly flexible in the other, instantaneous action at right angles to the axis of elasticity produces a deflection which is to that of accumulative action as $\sqrt{4}$ to 1, whilst the tendencies of fracture are as 4 to 1. But, should any case occur where the law of elasticity follows an extremely high power of the deflection, then the singular result will follow, that the deflections are the same whether the force be exerted from the initial state or the state of load, but that the tendency to fracture will be immensely greater in the former case than in the latter.

4. In producing the fracture of natural substances, which all depart from the law of perfect elasticity as we approach the limit of fracture, the ratio of the effects of instantaneous and accumulative action will vary with the nature of the substances, never being less, for elastic bodies, than 2 to 1, nor, for flexible, than 4 to 1, and more usually approaching 3 or 4 to 1 for the former case, and 5 or 6 to 1 for the latter.

5. Let a vase or conduit be acted upon by a load which is alone insufficient to break it; and let this load be partly balanced by a small exterior force. Should the great interior force suddenly cease, the small exterior action may crush the vase or conduit inward, its energy in such case being the sum of the interior and exterior forces.

6. Should the interior force be a vibration of the kind already explained, and should the exterior action be extremely feeble, and act on a very great mass, this extremely feeble action may crush the vase inward, with a power that shall exceed, in any degree, the enormous action of the interior and explosive vibrations. The comparison of the interior and exterior actions is best effected, in this case, by finding the modulus of elasticity of a material spring that shall coincide most nearly in effect with the interior tremor. For, putting $e$ and $e'$ respectively for the modulus of the spring and of the support, and $\sigma$ and $\sigma'$ for the deflections resulting from the tremor acting alone, and the reaction as it does act, we have $\frac{s'}{\sigma} = \sqrt{\frac{e}{e'}}$, or, in other words, the deflection produced by the reaction, is to the deflection that would be produced by the interior tremor alone, in the inverse proportion of the square roots of the moduli of tremor and support.

7. Combining what is here said with the known laws of fluids moving in pipes, and whereby they necessarily produce hydraulic shocks, it follows that any vessel connected with such a train of pipes, and plunged at some little depth in a considerable mass of water, or other heavy fluid, will occasionally be subject to a crushing and exterior force vastly greater than the interior strain due to the constant head of fluid.

To investigate these results, let us commence with the very simple case of a mass $m$, urged by two moving forces $f$ and $-f'$, which tend to deflect it from a point of initial repose. The vis viva in this case will be $\int_0^s \{fds - f'ds\}$, where $s$ is the distance through which the mass has moved; and as this vis viva must be zero when the mass has attained its limit of deflection, we have for that case

$$\int_0^s \{fds\} = \int_0^s \{f'ds\} \qquad (1)$$

Now it is clear that if the forces $f$ and $-f'$ acted only at the position of load, we should have simply

$$f = f'; \qquad (2)$$

and consequently if we express by $\sigma$ and $\sigma'$ the deflections occasioned by an instantaneous and an accumulated load, resulting from the action of $f - f'$ when commencing at the initial position, and the position of load, we must determine the first from equation (1), and the second from equation (2).

The case that we have first to consider, is when the force $-f'$ arises from the resistance $es$ of an elastic support, whilst $f$, as ordinarily happens, is a constant load. The equations (1 and 2) then become

$$2f\sigma = e\sigma^2$$
$$f = e\sigma'$$

whence we deduce the second of the results already mentioned, namely that $\sigma = 2\sigma'$.

When the elasticity is imperfect, we may reason by observing, first, that $s$ represents the abscissa, and $e\,s$ the ordinate $y$, of the curve of elasticity, or the curve which expresses the relation between the deflection and resistance; and, secondly, that the integral $\int fds$ is equivalent to the area of this curve. Now an area corresponding to an abscissa $s$ is equal to $s$ multiplied into the mean ordinate $y'$; and when the curve deflects inward from the tangent, and turns towards the abscissa, as the curve in question does when the elasticity is imperfect, it is manifest that the mean ordinate does not stand in the middle of the total abscissa, but at a point which is nearer to the origin: in other words $\sigma$ being the total abscissa, and $\sigma'$ that belonging to the mean ordinate, we shall have in the equation $\sigma = m\sigma'$, $m$ exceeding 2. But referring to our equations (1) and (2), we observe them to become in this case

$$f\sigma = y'\sigma$$

and

$$f = y'';$$

which gives $y' = y''$, or the mean ordinate of force to the deflection $\sigma$, equal to the extreme ordinate of force to the deflection $\sigma'$. And we have already seen that in the equation $\sigma = m\,\sigma'$, $m$ exceeds 2; whence we conclude so much of the fourth proposition as relates to bodies imperfectly elastic in the direction of the load.

To demonstrate the third proposition, let a cord of inconsiderable mass, half length $l$, and modulus $e$, be drawn into a horizontal position by the action of an inconsiderable stretching force. Applying to the centre of this cord a force that deflects it through a space $s$, the half length after deflection will be $\sqrt{l^2 + s^2}$; the extension $\frac{1}{2}\,\frac{s^2}{l}$; and the tension in the direction of each branch of

the cord, equal to $\frac{1}{2}\,\frac{es^2}{l}$. The resistance opposed to the deflecting force is therefore $\frac{es^3}{l^2}$; or $f$ must be regarded as constant, and $f'$ as equal to $\frac{es^3}{l^2}$; assumptions that reduce the equations (1) and (2) to

$$f\,\sigma = \tfrac{1}{4}\,\frac{e\,\sigma^4}{l^2}$$

$$f = \frac{e\,\sigma'^3}{l^2};$$

which give $\sigma = \sigma'\,\sqrt[3]{4}$, or the deflection from the initial position to the deflection of load, as $\sqrt[3]{4}$ to 1.

When the deflection, produced in either way, attains the limit of rupture $\sigma''$, we have $\sigma = \sigma' = \sigma''$; and denoting by $f_\backslash$ the force that would produce this deflection when acting from the initial position, and by $f$ the force that would produce the same as a deflection of load, the equation $\sigma = \sigma'$ gives

$$\frac{4f_\backslash}{f} = 1$$

or $f = 4f_\backslash$; or, the force which produces rupture when acting from the position of load, is to the force which produces fracture when acting from the initial position, as 4 to 1; an imperfection in the elasticity increasing this proportion, as in the former case.

And observing that when the law of elasticity is as the $m^{\text{th}}$ power of the deflection, these equations become

$$f\,\sigma = \tfrac{1}{m}\,\frac{e\,\sigma^{m+1}}{l^2}$$

$$f = \frac{e\,\sigma'^{\,m}}{l^2}$$

$$\frac{mf_\backslash}{f} = 1\,,$$

we deduce for the case when $m$ is infinite, the conclusion of the third proposition in regard to that case.

The application of this theory to the strains occasioned by a column of fluid moving through a pipe, and subject to checks, and separation into distinct columns, will be immediately seen. The most powerful of the interior strains of this class must be due to those blows of the hydraulic ram which occur, both

on the first letting on of the water, and on the suddenly closing a stop-cock when the fluid is in one of those pulses of attenuation which occur in the motion of fluids through long pipes, obstructed by enclosed air, and rendered irregular by branches.

How very powerful these blows are is well known to the engineer, and we have now only to show under what circumstances they may be reversed in direction, and vastly increased in intensity.

The inverted direction will occur whenever the pipe, pressed from without by the atmosphere, and any large mass of fluid, as the water of a well or pit through which it may pass, is left unsupported within by the sudden separation and contraction of the fluid which follows an hydraulic blow. Such inversion will always occasion a strain from without, more powerful than the internal strain that produced it, but the severest strains of this kind will occur when the original force is such as we have termed an interior vibration of immense momentum and infinitely little extent, compared with the motions and mass of the support.

It is true that such a case is never practically attained, and that it far more usually happens that the internal strain has an extent of motion approaching to, or equalling that of the support; in which case the reversed pressure becomes the sum of the interior and exterior forces; but as this forms one limit of the practical action, so may the explosive forces that we have described be said to form the other; and I shall therefore consider it proper to give their theory as connected with the subject under discussion. The very instance, indeed, which Dr. Hare describes, and that which fell under my own observation, approach this class, as will be evident in the former case, from considering that a pipe passing through a large metal reservoir, or chamber, might have the lateral pulses propagated in the water it conveyed, small with regard to the expansion which the chamber was capable of enduring; and this to the greater extent, as we speak not of the total lateral motion of the pulse, but of that lateral expansion which it would undergo during its extremely rapid transit through the chamber.

Such forces would nearly resemble those internal explosions that we assume; and as the sides of the reservoir in the case alluded to were of copper, and of no great thickness, they will approach the remaining conditions of the problem, partaking of the nature both of flexible and of elastic bodies, and having little

mass compared with the column of fluid within the pipe, or the body of fluid without.

The instance of explosive forces on which I shall found their theory will be readily understood. Conceive an elastic lamina $a$, of inconsiderable mass, and small modulus $e'$ to be placed between two masses $m$ and $m'$, both large compared to $a$, but of which $m'$ infinitely exceeds $m$. Let $m$ be attached to a spring $b$ that is retained at a distance $s'$ from the point of repose, very inconsiderable with regard to the deflections of $a$, but which, from the amount of the modulus $e$ of this second elastic body, will occasion, when the retaining power is removed, an immense pressure on $m$, $a$, and $m'$. Further, conceive $m'$ subject to a small constant force that urges it in a contrary direction to this pressure.

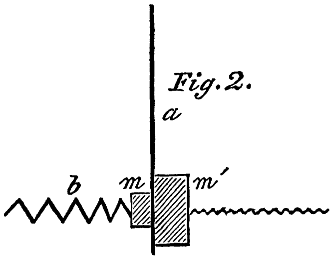


*Fig. 2.*

The magnitude of $e$'s and $m'$ will allow us to consider these as the only elements concerned in the generation of the first motions, and thus when $b$ has attained the position of repose, or that where it has no elastic power, the vis viva generated will be $e\,\sigma^2$; and as the motion of the mass $m'$, attached to $b$, is now retarded, $m'$ and $m$ will separate; and the latter, after being urged to a distance which, on account of the minuteness of the resisting force, is considerable, will return to impinge on $a$, with the vis viva $e\,\sigma^2$; and if $b$ has been removed in the interval, this force must be destroyed by the sole resistance of $a$. But the vis viva generated by $a$ in traversing a distance $\sigma'$ is equal to $e'\,\sigma'^2$; from which it follows that when the returning strain occasioned by $m'$ has been wholly destroyed, we have

$$e\,\sigma^2 = e'\,\sigma'^2$$

or

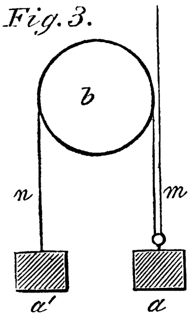
$$\frac{\sigma'}{\sigma} = \sqrt{\frac{e}{e'}}$$

as asserted in the sixth proposition. The use of the mass $m'$ in this investigation will be readily seen by observing that it makes the velocity of the motion, when $b$ acts against $a$ alone, quite independent of the mass of the latter; so that $a$ will stop when $b$ has expanded through the extent of a vibration, and the deflection thus attained will be $\sigma$; or, in other words, the case assumed is that of a heavy and powerful spring, acting against a light and weak one.

I shall conclude these remarks with illustrations derived from two obvious and simple experiments.

Thus an examination of the relative power which different filaments possess in resisting instantaneous and accumulative loads may be conducted as follows:—

Fig. 3.




Suspend from a thread ($m$) of the substance examined, a weight ($a$) that is insufficient to cause rupture: let a weight ($a'$) exceedingly nearly equal to ($a$) support the latter through the medium afforded by the fine thread ($n$) that passes over the very delicate pulley ($b$). In this state of things it is evident that ($m$) will merely sustain such tension as suffices to draw it into a rectilinear position; or, in other words, to bring it into the state where elasticity begins to be exerted. Now, by the rapid lateral motion of some sharp edge, as that of a razor, divide ($n$); and thus transfer, instantaneously, the whole tension of ($a$) upon the thread ($m$). The load, in this case, is that which we have termed instantaneous, and, in consequence, rupture will take place under the action of a weight considerably less than if the pressure had been made accumulative, as, by gradually raising ($a'$) until ($m$) had obtained the deflection of load. In the instance of a thread ot fine cotton, the proportion of the weights that caused rupture in the two cases was as 1 to 3.

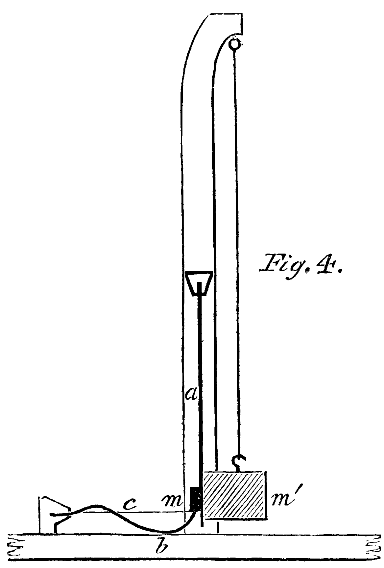



Fig. 4.

Experiments on the restricted vibrations which we termed explosive, and which last occupied our attention, may be conducted by means of the very case that we chose for investigating the theory of such forces. A massive body ($m'$), suspended by a thread of considerable length, and allowed to rest in its state of equilibrium against the elastic rod that is to be broken, will supply both the mass ($m'$) of the theory, and the weak and nearly constant force that urged it; whilst the smaller mass ($m$), set into motion by suddenly cutting the cord ($c$), will continue to press against ($a$) until the spring ($b$) has nearly attained its point of repose, when, no longer supported by

friction, the weight of $(m)$ will detach it from $(a)$, and bring $(b)$ into such a position as to prevent its impeding the reaction by which $(a)$ is broken.

The sides of the reservoir, in the case which led to this investigation, we have assimilated to $(a)$; the brief but powerful hydraulic concussions of the internal fluid, to the action of the spring $(b)$, and the effect of the external water in which we suppose the reservoir plunged, to that of the mass $(m')$; and I need merely add, that were the cases perfectly parallel, our formula proves that the tendency to rupture by the reaction may amount to a very large multiple of that resulting from the mere explosive strain when the exterior pressure is removed.